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2009 J. Phys. A: Math. Theor. 42 332002

(http://iopscience.iop.org/1751-8121/42/33/332002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 332002 (7pp) doi:10.1088/1751-8113/42/33/332002

FAST TRACK COMMUNICATION

An inverse Born approximation for the general nonlinear Schrödinger operator on the line

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Received 27 May 2009, in final form 1 July 2009
Published 22 July 2009
Online at stacks.iop.org/JPhysA/42/332002

Abstract
This work deals with the inverse scattering problems for the one-dimensional Schrödinger equation of the most general form

$$-u'' + F(x, u) = k^2 u,$$

with general nonlinearity F and with a real parameter k. Under some assumptions on F we prove that all singularities and jumps of $F(x, u_0)$, where $u_0(x, k) = e^{ikx}$, can be recovered by the reflection coefficient with arbitrary large k.

PACS numbers: 02.30.Zz, 03.65.Nk, 02.30.Nw
Mathematics Subject Classification: 35P25, 35R30

1. Introduction

Consider the nonlinear Schrödinger equation

$$H_F u(x) := -u''(x) + F(x, u) = k^2 u(x), \quad k \in \mathbb{R}, \quad k \neq 0, \tag{1.1}$$

where F satisfies the following conditions:

$$\begin{aligned} &(i) \quad \overline{F(x, \overline{u})} = F(x, u), \\ &(ii) \quad F(x, u_0+v) = u_0^\delta h_0(x) + \left(u_0^\alpha h_1(x) + \tilde{h}_1(x)\right) v + u_0^\beta h_2(x)\overline{v} + R|v|^\gamma, \end{aligned} \tag{1.2}$$

where $u_0 = e^{ikx}$, functions $h_0, h_1, \tilde{h}_1, h_2$ are real valued and belong to $L^1(\mathbb{R})$, $\delta, \alpha, \beta, \gamma$ are real parameters and $\delta \neq -1, \gamma > 1$. In particular, $F(x, u_0) = u_0^\delta h_0(x)$. Real-valued function R in (1.2) may depend on x, k and v. We assume then that R satisfies the condition

$$|R(x, k, v)| \leqslant h_3(x) \tag{1.3}$$

with h_3 from $L^1(\mathbb{R})$ and the Lipschitz condition

$$|R(x, k, v_1) - R(x, k, v_2)| \leqslant \tilde{h}_3(x)|v_1 - v_2| \tag{1.4}$$

with $\tilde{h}_3$ from $L^1(\mathbb{R})$. For $u \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, we understand equation (1.1) in the sense of Schwartz distributions.

Equation (1.1) with concrete F appears quite naturally in applications. It includes the linear case and the basic nonlinearities of cubic and cubic-quintic type. These equations can be met in nonlinear optics in the context of Kerr-like nonlinear dielectric film, see [8, 9, 13, 14]. This is an interesting example of more general nonlinearity, which is called the 'Morse oscillator',

$$-u''(x) - \mu\left(\mathrm{e}^{-a|u|} - \mathrm{e}^{-2a|u|}\right) = k^2 u(x).$$

Some modifications of the latter equation are used in theoretical chemistry to describe the photo-dissociation of molecules. Especially the classical Morse oscillator driven by a sinusoidal force has been invoked in studies of stochastic excitation. Here μ is dissociation energy and a is a range parameter.

In scattering theory one considers the scattering solutions to equation (1.1), that is the solutions of the form

$$u(x,k) = u_0(x,k) + u_{sc}(x,k),$$

where $u_0(x,k) = \mathrm{e}^{\mathrm{i}kx}$ is the incident wave and $u_{sc}(x,k)$ is the scattered wave. These solutions are the unique solutions of the Lippmann–Schwinger equation

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} + \frac{1}{2\mathrm{i}|k|}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}|k||x-y|} F(y,u)\,\mathrm{d}y.$$

Setting $u(x,k) = \overline{u(x,-k)}$ for $k < 0$ and using (i) from (1.2) we can consider the following integral equation:

$$u(x,k) = \mathrm{e}^{\mathrm{i}kx} + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k|x-y|} F(y,u)\,\mathrm{d}y \tag{1.5}$$

for all $k \neq 0$.

We need some properties of the solutions $u(x,k)$ to equation (1.5). Let us introduce the following sequence:

$$u_j(x,k) = \mathrm{e}^{\mathrm{i}kx} + \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k|x-y|} F(y,u_{j-1})\,\mathrm{d}y, \tag{1.6}$$

where $j = 1, 2, \ldots$.

Lemma 1.1. *If*

$$k \geqslant k_0 := \max\left(\gamma^{\frac{1}{\gamma-1}}\|h_0\|_1, \|h_1\|_1 + \|\tilde{h}_1\|_1 + \|h_2\|_1 + \|h_3\|_1 + \|\tilde{h}_3\|_1\right) + 1,$$

then there exists a unique solution $u(x,k) \in L^\infty(\mathbb{R})$ to (1.5) of the form $u = u_0 + u_{sc}$ such that, uniformly in $x \in \mathbb{R}$ and $k \geqslant k_0$,

$$u(x,k) = \lim_{j\to\infty} u_j(x,k).$$

Moreover, for each $j = 0, 1, 2, \ldots,$

$$\|u - u_j\|_\infty \leqslant \frac{c_j}{k^{j+1}}. \tag{1.7}$$

Proof. For each $j = 0, 1, 2, \ldots,$ equation (1.6) implies by induction that

$$\|u_{j+1}\|_\infty \leqslant 1 + \frac{\|h_0\|_1}{2k}\sum_{l=0}^{j}\left(\frac{A}{2k}\right)^l$$

where $A = \|h_1\|_1 + \|\tilde{h}_1\|_1 + \|h_2\|_1 + \|h_3\|_1$. Hence, if we choose $k > A$ then we obtain

$$\|u_{j+1}\|_\infty \leqslant 1 + \frac{\|h_0\|_1}{k}, \qquad j = 0, 1, 2, \ldots$$

Using now the latter inequality and (1.2)–(1.4) we obtain

$$\|u_{j+1} - u_j\|_\infty \leqslant \frac{1}{2k}\int_{-\infty}^{\infty}\left(|h_1| + |\tilde{h_1}| + |h_2| + \left(\frac{\|h_0\|_1}{k}\right)^\gamma |\tilde{h_3}|\right.$$
$$\left. + \gamma\left(\frac{\|h_0\|_1}{k}\right)^{\gamma-1}|h_3|\right)\mathrm{d}y\|u_j - u_{j-1}\|_\infty.$$

If we choose first $k \geqslant \gamma^{\frac{1}{\gamma-1}}\|h_0\|_1$ then we obtain

$$\|u_{j+1} - u_j\|_\infty \leqslant \frac{1}{2k}\int_{-\infty}^{\infty}(|h_1| + |\tilde{h_1}| + |h_2| + |\tilde{h_3}| + |h_3|)\,\mathrm{d}y\|u_j - u_{j-1}\|_\infty \tag{1.8}$$

and uniformly with respect to $k \geqslant k_0$ (where k_0 is as in the lemma)

$$\|u_{j+1} - u_j\|_\infty \leqslant \tfrac{1}{2}\|u_j - u_{j-1}\|_\infty.$$

This inequality allows us to conclude that the sequence $\{u_j\}_{j=0}^\infty$ is a Cauchy sequence in the space $L^\infty(\mathbb{R})$. Thus, the first part of this lemma is proved. But inequality (1.7) follows immediately from (1.8). Lemma 1.1 is completely proved now.

The results of lemma 1.1 imply that for fixed positive $k \geqslant k_0$ the solution $u(x, k)$ admits for $x \to -\infty$ the asymptotic representation

$$u(x, k) = \mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx}b(k) + o(1),$$

where

$$b(k) = \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}ky}F(y, u)\,\mathrm{d}y, \qquad k \geqslant k_0. \tag{1.9}$$

Note that $b(k) = \overline{b(-k)}$. We consider $b(k)$ as our scattering data and set $b(k) = 0$ for $|k| < k_0$. The inverse problem that is considered here is to extract information about $h_0(x)$ given the scattering data, $b(k)$ for all $k \geqslant k_0$. We will investigate an approximate method to recover the discontinuities of the function $h_0(x)$. This method is called the Born approximation.

Definition (1.9) of $b(k)$ and property (1.7) of the solutions $u(x, k)$ allow us to conclude that for large k

$$b(k) \approx \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k(1+\delta)y}h_0(y)\,\mathrm{d}y = \frac{\sqrt{2\pi}}{2\mathrm{i}k}F(h_0)(k(1+\delta)),$$

where h_0 is the function which is defined in (1.2) and F designates the Fourier transform

$$(Ff)(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}ky}f(y)\,\mathrm{d}y.$$

The inverse Fourier transform is thus defined by

$$(F^{-1}f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}f(k)\,\mathrm{d}k.$$

The latter formulae justify the following definition.

□

Definition 1. *The inverse Born approximation $q_B(x)$ for the Schrödinger equation (1.1) is defined by*

$$q_B(x) = F^{-1}\left(\frac{2\mathrm{i}kb\left(\frac{k}{1+\delta}\right)}{\sqrt{2\pi}(1+\delta)}\right)(x). \tag{1.10}$$

Note that this equality must be considered in the sense of tempered distributions.

There is the interest in the unique (or full) recovery of the unknown function(s), see Gelfand–Levitan–Marchenko approach [1, 3–7, 10] in the linear case and the works of Strauss [20], Weder [22–24], Tao [21] and Aktosun *et al* [2] in various nonlinear settings.

Some works are concerned with the partial recovery of the unknown functions. Several authors (we restricted this list only to one-dimensional case) have proved that an approximate method known as the inverse Born approximation is able to recover all jumps and singularities of the unknown function in the linear settings from limited data, see, e.g., [11, 12, 15, 18, 19] and the references therein. In one dimension, this approach requires less than half the data, roughly speaking, compared to the attempt of full recovery. Our line of research is this very approach. After providing the first application of the Born approximation in the nonlinear case (see [16, 17]) for particular type of nonlinearity F, we extend it in this communication to very general nonlinearities, namely, those of forms (1.2)–(1.4).

The main goal of this communication is to prove the following theorem.

Theorem 1. *Assume that the functions $h_0, h_1, \tilde{h}_1, h_2, h_3, \tilde{h}_3$ from (1.2)–(1.4) are real valued and belong to $L^1(\mathbb{R})$. Then the difference $q_B(x) - h_0(x)$ is continuous on the line.*

2. Proof of the main theorem

The convergence of the solution sequence $\{u_j\}_{j=0}^{\infty}$ from lemma 1.1 gives rise to the data sequence

$$b_j(k) = \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky} F(y, u_j)\,\mathrm{d}y, \qquad j = 0, 1, 2, \ldots,$$

such that $b(k) = \lim_{j\to\infty} b_j(k)$ uniformly and the Born sequence $q_{B,j}(x)$

$$q_{B,j}(x) = \frac{\mathrm{i}}{\pi(1+\delta)}\int_{-\infty}^{\infty} kb_j\left(\frac{k}{1+\delta}\right)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k, \quad j = 0, 1, 2, \ldots, \tag{2.1}$$

of our main interest. Here the convergence,

$$q_B(x) = \lim_{j\to\infty} q_{B,j}(x),$$

can be regarded in the sense of distributions.

Lemma 2.1. *Under the same assumptions as in theorem 1.1*

$$q_{B,0}(x) = h_0(x) \quad (\mathrm{mod}\ C^{\infty}(\mathbb{R})).$$

Proof. It is the same as lemma 4 in [17]. □

Lemma 2.2. *Under the same assumptions as in theorem 1.1 for each $j = 0, 1, 2, \ldots,$*

$$|b(k) - b_j(k)| \leqslant \frac{c_j}{k^{j+2}}, \qquad k \geqslant k_0. \tag{2.2}$$

Proof. Since $u(x,k) = u_0(x,k) + u_{sc}(x,k)$, $u_j(x,k) = u_0(x,k) + u_{sc}^{(j)}(x,k)$ and

$$|b(k) - b_j(k)| \leqslant \frac{1}{2k}\int_{-\infty}^{\infty} |F(y,u) - F(y,u_j)|\,\mathrm{d}y, \qquad k \geqslant k_0,$$

we may conclude from (1.2)–(1.4) and from the proof of lemma 1.1 that

$$|b(k) - b_j(k)| \leqslant \frac{1}{2k}\left(\|h_1\|_1 + \|\tilde{h}_1\|_1 + \|h_2\|_1 + \|h_3\|_1 + \|\tilde{h}_3\|_1\right)\|u - u_j\|_\infty \leqslant \frac{c_j}{k^{j+2}}.$$

It proves this lemma.

□

It follows from lemma 2.2 that

$$b(k) = b_1(k) + O\left(\frac{1}{k^3}\right), \qquad k \geqslant k_0.$$

This is enough for us to conclude (see (1.10) and (2.1)) that

$$q_B(x) - q_{B,1}(x) \in C^\beta(\mathbb{R}) \tag{2.3}$$

for any $0 < \beta < 1$, where C^β denotes Zygmund-Hölder space on the line.

The smoothness of the term $q_{B,1}$ can only be estimated after we can write it in a more explicit form.

Lemma 2.3. *Under the same assumptions as in theorem 1.1*

$$q_{B,1}(x) = h_0(x) + q_1(x) \qquad \left(\mathrm{mod}\, H^s(\mathbb{R}) + \mathrm{mod}\, C^\infty(\mathbb{R})\right)$$

for any $s < \gamma - \frac{1}{2}$, where $H^s(\mathbb{R})$ denotes Sobolev space, and the first nonlinear term q_1 has a precise form

$$\begin{aligned} q_1(x) = \frac{1+\delta}{4}\int_{-\infty}^{\infty} h_0(z)\,\mathrm{d}z \int_{-\infty}^{\infty}\Bigg(& h_1(y)\,\mathrm{sgn}\left(\frac{1+\alpha}{1+\delta}y + \frac{\delta}{1+\delta}z + \frac{|y-z|}{1+\delta} - x\right)\\ &+\tilde{h}_1(y)\,\mathrm{sgn}\left(\frac{1}{1+\delta}y + \frac{\delta}{1+\delta}z + \frac{|y-z|}{1+\delta} - x\right)\\ &+h_2(y)\,\mathrm{sgn}\left(\frac{1+\beta}{1+\delta}y - \frac{\delta}{1+\delta}z - \frac{|y-z|}{1+\delta} - x\right)\Bigg)\,\mathrm{d}y. \end{aligned} \tag{2.4}$$

Proof. Due to representations (1.2) and (2.1), lemma 2.1 and the definition of $u_{sc}^{(1)} = u_1 - u_0$ the term $q_{B,1}$ can be rewritten as

$$\begin{aligned} q_{B,1}(x) = h_0(x) &+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\int_{-\infty}^{\infty}\left(\mathrm{e}^{\mathrm{i}\frac{k(1+\alpha)y}{1+\delta}}h_1(y) + \mathrm{e}^{\mathrm{i}\frac{ky}{1+\delta}}\tilde{h}_1(y)\right)u_{sc}^{(1)}\,\mathrm{d}y\,\mathrm{d}k\\ &+\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{k(1+\beta)y}{1+\delta}}h_2(y)\overline{u_{sc}^{(1)}}\,\mathrm{d}y\,\mathrm{d}k\\ &+\frac{1}{2\pi}\int_{|k|\geqslant k_0}\mathrm{e}^{-\mathrm{i}kx}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{ky}{1+\delta}}R\left|u_{sc}^{(1)}\right|^\gamma\,\mathrm{d}y\,\mathrm{d}k \qquad (\mathrm{mod}\, C^\infty(\mathbb{R})), \end{aligned} \tag{2.5}$$

where $u_{sc}^{(1)}$ and R are taken with the variables $\left(\frac{k}{1+\delta}, y\right)$ and $\left(\frac{k}{1+\delta}, y, u_{sc}^{(1)}\right)$, respectively.

Since condition (1.3) is satisfied and $\left|u_{sc}^{(1)}\right| \leqslant \frac{c}{|k|}$ we can conclude that the last integral in (2.5) is a function of $x \in \mathbb{R}$ from Sobolev space $H^s(\mathbb{R})$ for any $s < \gamma - \frac{1}{2}$. Further, substituting the exact form of $u_{sc}^{(1)}$ (see (1.6)) in the first and the second integrals of (2.5), we obtain (2.4) (see the proof of lemma 11 in [17] for details). Thus, the lemma is proved.

□

Corollary 1. *q_1 is a bounded continuous function.*

Now we are ready to prove the main theorem.

Proof of theorem 1. Since

$$q_B(x) - h_0(x) = (q_B(x) - q_{B,1}(x)) + (q_{B,1}(x) - h_0(x)),$$

then the theorem follows immediately from (2.3), lemma 2.3, its corollary 1 and the embedding theorem for Sobolev spaces.

□

Remark 1. This theorem allows us to conclude that all jumps and singularities of the unknown function h_0 (which might be considered as the characterization of nonlinearity F) can be obtained exactly by the Born approximation. In particular, for $h_0(x)$ being the characteristic function of the unknown interval on the line this interval is uniquely determined by this scattering data.

Acknowledgment

This work was supported by the Academy of Finland (application number 213476, Finnish Programme for Centres of Excellence in Research 2006–2011).

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